MATH 2060 TUTO 10

§8.1 6. Show that $\lim(\text{Arctan } nx) = (\pi/2)\text{sgn } x$ for $x \in \mathbb{R}$.

Ans: (Pointwise convergence)

Let $g_n(x) = \text{Arctan}(nx)$

- If $x = 0$, then $g_n(0) = 0 \to 0$ as $n \to \infty$
- If $x > 0$, then $g_n(x) = \text{Arctan}(nx) \to \frac{\pi}{2}$ as $n \to \infty$
- If $x < 0$, then $g_n(x) = \text{Arctan}(nx) \to -\frac{\pi}{2}$ as $n \to \infty$

Hence $\lim(g_n(x)) = (\frac{\pi}{2})\,\text{sgn}\, x \quad \forall x \in \mathbb{R}$

$=: g(x)$

16. Show that if $a > 0$, then the convergence of the sequence in Exercise 6 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $(0, \infty)$.

Ans: Let $a > 0$. Then

$$\|g_n - g\|_{[a,\infty)} = \sup_{x \in [a,\infty)} \left|\text{Arctan}(nx) - \frac{\pi}{2}\right| = \frac{\pi}{2} - \text{Arctan}(na)$$

$$\Rightarrow \lim_{n\to\infty} \|g_n - g\|_{[a,\infty)} = \lim_{n\to\infty}\left(\frac{\pi}{2} - \text{Arctan}(na)\right) = 0$$

So $g_n \rightrightarrows g$ on $[a, \infty)$

OTOH, $\forall a > 0$

$$\|g_n - g\|_{(0,\infty)} \geq \|g_n - g\|_{[a,\infty)} = \frac{\pi}{2} - \text{Arctan}(na)$$

Letting $a \to 0^+$, we have

$$\|g_n - g\|_{(0,\infty)} \geq \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

So, (g_n) does not converge (to g) uniformly on $(0, \infty)$.

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Problem 11.6 Determine whether the sequence $\{f_n\}$ converges uniformly on D.

a) $f_n(x) = \dfrac{1}{1+(nx-1)^2}$ $\quad D = [0,1]$

b) $f_n(x) = nx^n(1-x)$ $\quad D = [0,1]$

c) $f_n(x) = \arctan\left(\dfrac{2x}{x^2+n^3}\right)$ $\quad D = \mathbb{R}$

a) Note $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{1}{1+(nx-1)^2} = \left\{ \begin{array}{ll} 0 & x \in (0,1] \\ \frac{1}{2} & x = 0 . \end{array} \right\} =: f(x)$

(We can actually conclude here that the convergence is not uniform, for otherwise the limit fun is also cts)

Lemma 8.1.5 (f_n) does not converge uniformly on A to f iff $\exists\, \varepsilon_0 > 0$, $\exists$ subseq (f_{n_k}) of (f_n), $\exists\, (x_k) \in A$ s.t

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}.$$

Take $\varepsilon_0 = 1$ and $x_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$.

Then $|f_n(x_n) - f(x_n)| = |f_n(x_n) - 0| = 1 \geq \varepsilon_0$ $\quad \forall n \in \mathbb{N}$.

Hence (f_n) does not converge uniformly on $[0,1]$ //

Problem 11.6 Determine whether the sequence $\{f_n\}$ converges uniformly on D.

a) $f_n(x) = \dfrac{1}{1+(nx-1)^2}$ $\quad D = [0,1]$

b) $f_n(x) = nx^n(1-x)$ $\quad D = [0,1]$

c) $f_n(x) = \arctan\left(\dfrac{2x}{x^2+n^3}\right)$ $\quad D = \mathbb{R}$

b) Note $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx^n(1-x) = 0 =: f(x) \quad \forall x \in [0,1]$.

$(\|f_n - f\|_{[0,1]} = \ ? \ )$

$\forall n \geq 2$,

$$f_n'(x) = n^2 x^{n-1}(1-x) + nx^n(-1) = nx^{n-1}(n-(n+1)x)$$

So $\quad f_n'(x) = 0 \iff x = 0$ or $x = \dfrac{n}{n+1}$.

Then f_n attains max on $[0,1]$ at

$x = 0$, $\quad x = 1$ $\quad$ or $\quad x = \dfrac{n}{n+1}$

$f_n(0) = 0$ $\quad f_n(1) = 0$ $\quad f_n\left(\dfrac{n}{n+1}\right) = \left(\dfrac{n}{n+1}\right)^{n+1}$

Thus $\|f_n - f\|_{[0,1]} = f_n\left(\dfrac{n}{n+1}\right) = \left(\dfrac{n}{n+1}\right)^{n+1}$

$$\Rightarrow \lim_{n\to\infty} \|f_n - f\|_{[0,1]} = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)^{n+1} = \frac{1}{e} \neq 0$$

Therefore (f_n) does not converge uniformly on $[0,1]$

Problem 11.6 Determine whether the sequence $\{f_n\}$ converges uniformly on D.

a) $f_n(x) = \dfrac{1}{1+(nx-1)^2}$ $\qquad D = [0,1]$

b) $f_n(x) = nx^n(1-x)$ $\qquad D = [0,1]$

c) $f_n(x) = \arctan\left(\dfrac{2x}{x^2+n^3}\right)$ $\qquad D = \mathbb{R}$

c) Note $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \arctan\left(\frac{2x}{x^2+n^3}\right) = 0 =: f(x) \quad \forall x \in \mathbb{R}$.

As in b), $f_n'(x) = \dfrac{1}{\left(\frac{2x}{x^2+n^3}\right)^2+1} \cdot \dfrac{2(x^2+n^3)-2x(2x)}{(x^2+n^3)^2}$

$$= \frac{2n^3-2x^2}{(x^2+n^3)^2+4x^2}$$

$\Rightarrow f_n'(x) = 0 \iff x = \pm n\sqrt{n}$

Since $|f_n(x)| \to 0$ as $x \to \pm\infty$, we have
$\sup\{|f_n(x) - 0| : x \in \mathbb{R}\} = |f_n(\pm n\sqrt{n})| = \arctan\left(\frac{1}{n\sqrt{n}}\right)$

Thus $\|f_n - f\|_{\mathbb{R}} = \arctan\left(\frac{1}{n\sqrt{n}}\right) \longrightarrow 0$ as $n \to \infty$

Hence (f_n) converges uniformly on $\mathbb{R}$ to $f \equiv 0$ //

§8.2 12. Show that $\lim \int_1^2 e^{-nx^2}\,dx = 0$.

Ans: Note $\lim_{n\to\infty} e^{-nx^2} = 0 \quad \forall\, x > 0$

If we can show that the convergence is unsform, then, by Thm 8.2.4,

$$\lim_{n\to\infty}\int_1^2 e^{-nx^2}dx = \int_1^2 0\,dx = 0$$

Indeed, $\forall\, x \in [1,2]$,

$$e^{nx^2} = 1 + (nx^2) + \frac{(nx^2)^2}{2!} + \cdots$$
$$\geq nx^2$$

$\Rightarrow \quad 0 \leq e^{-nx^2} \leq \frac{1}{nx^2} \leq \frac{1}{n}$ (Since $x \in [1,2]$)

So $e^{-nx^2} \rightrightarrows 0$ on $[1,2]$

By Thm 8.2.4

$$\lim_{n\to\infty}\int_1^2 e^{-nx^2}dx = \int_1^2 0\,dx = 0$$

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Ex Let f be a continuously differentiable function defined on (a,b) (i.e. f' is continuous). Consider the sequence (f_n):

$$f_n(x) = n\left(f\left(x+\frac{1}{n}\right) - f(x)\right)$$

Show that f_n converges uniformly to f' in any closed subinterval $[c,d]$ of (a,b).

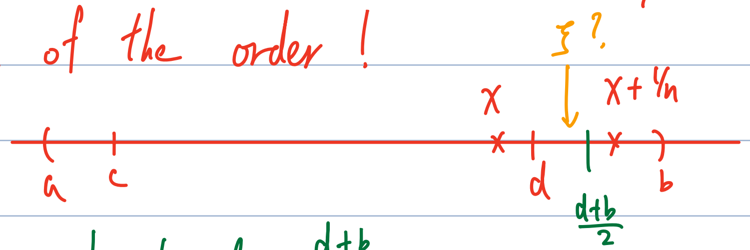
Pf: Idea: By MVT, $f_n(x) = \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n} - 0} = f'(\xi)$

for some $\xi = \xi(n,x) \in (x, x+\frac{1}{n})$

So $|f_n(x) - f'(x)| = \underbrace{|f'(\xi) - f'(x)|}_{}$

small since f' is cts, hence uniformly cts on any $[c,d] \subseteq (a,b)$

Be careful of the order!



Let $\varepsilon > 0$. Let $\ell = \frac{d+b}{2}$.

Since f' is cts and hence uniformly cts on $[c,\ell]$,

$\exists\ \delta > 0$ s.t. if $u, v \in [c,\ell]$ and $|u-v| < \delta$, then $|f'(u) - f'(v)| < \varepsilon$ (#)

Take $\delta' = \min\{\ell - d, \delta\} > 0$.

Choose $N \in \mathbb{N}$ s.t. $1/N < \delta'$ (so $x \in [c,d] \Rightarrow x + \frac{1}{N} \in [c,\ell]$)

$\forall\ x \in [c,d]$, $\forall\ n \geq N$, by MVT,

$\exists\ \xi = \xi(n,x) \in (x, x+\frac{1}{n}) \subseteq [c,\ell]$ s.t.

$$f'(\xi) = n\left(f(x+\tfrac{1}{n}) - f(x)\right) = f_n(x)$$

Since $x, \xi \in [c,\ell]$ and $|x - \xi| < \frac{1}{n} \leq \frac{1}{N} < \delta'$, we have

$$|f_n(x) - f'(x)| = |f'(\xi) - f'(x)| < \varepsilon \qquad \text{by (\#)}$$

Therefore $f_n \rightrightarrows f'$ on $[c,d]$ //